Transient Conduction in a Cylinder in an Infinite Conductive Medium with Contact Resistance

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Nomenclature

A	= ratio of volumetric heat capacities
h_c	= contact conductance, W/m^2-°C
$J_0, J_1; Y_0, Y_1$	= Bessel functions of the first kind
k_1, k_2	= thermal conductivities of media 1 and 2, W/m-°C
K	$= k_1/k_2$
Q_0	= heat generation rate, W/m^3
r	= distance from cylinder center, m
R	= outside radius of cylinder, m
t	= time, s
T	= temperature, °C
T_0	= initial temperature of cylinder
T_∞	= initial temperature of the infinite medium
α_1, α_2	= thermal diffusivities of media 1 and 2, m^2/s
θ_0	$= T_0 - T_\infty$
θ_1	$= T - T_\infty,\ 0 \le r \le R$
θ_2	$= T - T_\infty,\ R < r$
ϕ, ψ	= functions defined by Eqs. (10) and (9), respectively

Introduction

IN their classical treatise on heat conduction, Carslaw and Jaeger[1] give Laplace transform solutions for the temperatures in an infinite cylinder surrounded by an infinite conductive medium in perfect thermal contact. In a recent study[2] of thermal stresses in cylinders cooled by conduction, a solution of this problem with thermal contact resistance was derived. It is the purpose of the present Note to present this solution as well as that for a cylinder with heat generation, along with an effective scheme for numerical evaluation of the solutions. The results are of interest in extrusion processes where hot billets come in contact with cooler walls, in situations where highly conductive fluids are used as coolants, such as in fast breeder reactors, in electrically heated aerospace components, such as a plasma torch igniter, and in nozzles and nose cones.

Analysis

Consider first a cylinder initially at temperature T_0 placed in contact with a surrounding conductive medium at temperature T_∞. If we let $\theta(r,t) = T(r,t) - T_\infty$, the temperature in region 1 (the inside cylinder) must satisfy the equation

$$\frac{\alpha_1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta_1}{\partial r}\right) = \frac{\partial\theta_1}{\partial t}, \qquad 0 \le r \le R \tag{1}$$

with initial and boundary conditions

$$\theta_1(r,0) = \theta_0; \qquad \frac{\partial\theta_1}{\partial r}(0,t) = 0 \tag{2}$$

while the temperature in the outside medium is described by

$$\frac{\alpha_2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta_2}{\partial r}\right) = \frac{\partial\theta_2}{\partial t}, \qquad R \le r \le \infty \tag{3}$$

with initial and boundary conditions

$$\theta_2(r,0) = 0; \qquad \lim_{r\to\infty}\theta_2(r,t) = 0 \tag{4}$$

At the interface $r = R$, the temperatures must satisfy the conditions

$$k_1\frac{\partial\theta_1}{\partial r}\bigg|_{r=R} = k_2\frac{\partial\theta_2}{\partial r}\bigg|_{r=R} \tag{5}$$

and

$$\theta_1(R,t) - \theta_2(R,t) = \frac{-k_1}{h_c}\frac{\partial\theta_1}{\partial r}\bigg|_{r=R} \tag{6}$$

The equations are solved by Laplace transformation, with inverse transforms computed by contour integration,[1,3] to find

$$\frac{\theta_1(r,t)}{\theta_0} = \frac{4k_1k_2\alpha_2}{\pi^2 R}\int_0^\infty \frac{J_1(uR)J_0(ur)\exp(-\alpha_1u^2t)}{u^2[\psi^2(u)+\phi^2(u)]}\mathrm{d}u, \qquad 0<r<R \tag{7}$$

and

$$\frac{\theta_2(r,t)}{\theta_0} = \frac{2k_1\sqrt{\alpha_2}}{\pi}\int_0^\infty \frac{J_0(aur)\phi(u) - Y_0(aur)\psi(u)}{u[\psi^2(u)+\phi^2(u)]} \times J_1(uR)\exp(-\alpha_1u^2t)\mathrm{d}u, \qquad R<r<\infty \tag{8}$$

In Eqs. (7) and (8), $a = \sqrt{\alpha_1/\alpha_2}$,

$$\psi(u) = k_1\sqrt{\alpha_2}J_0(auR)J_1(uR) - k_2\sqrt{\alpha_1}J_0(uR)J_1(auR) + (k_1k_2\sqrt{\alpha_1}u/h_c)J_1(uR)J_1(auR) \tag{9}$$

and

$$\phi(u) = k_1\sqrt{\alpha_2}J_1(uR)Y_0(auR) - k_2\sqrt{\alpha_1}J_0(uR)Y_1(auR) + (k_1k_2\sqrt{\alpha_1}u/h_c)J_1(uR)Y_1(auR) \tag{10}$$

Note that the results given in Eqs. (7) and (8) reduce to those given by Carslaw and Jaeger[1] in the limit $h_c \to \infty$.

With uniform heat production Q_0 per unit time per unit volume in the inner cylinder, $0 \le r \le R$, and with initial

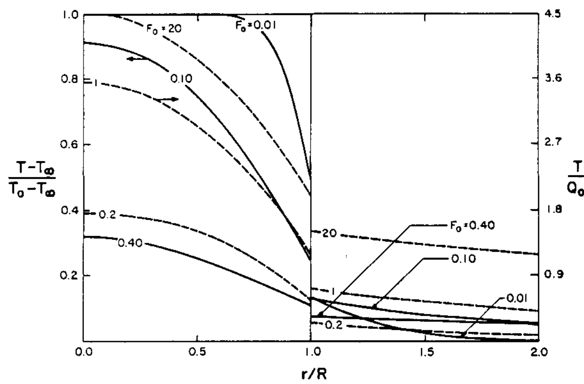

Fig. 1 Temperatures in an infinite cylinder bounded by an infinite conducting medium after a sudden change in medium temperature (—) and sudden initiation of heat generation (----). $Bi_c = 10$, $K = 0.1$, $A = 1$.

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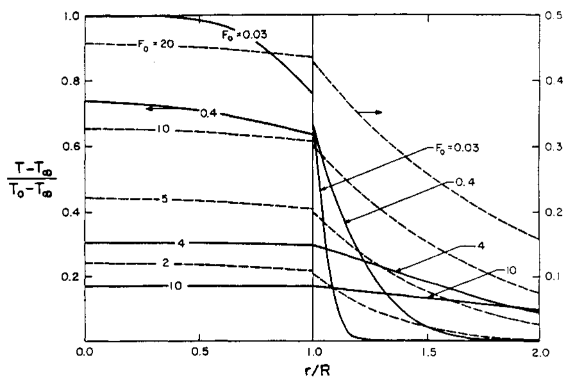

Fig. 2 Temperatures in an infinite cylinder bounded by an infinite conducting medium after a sudden change in medium temperature (—) and sudden initiation of heat generation (----). $Bi_c = 10$, $K = 10$, $A = 1$.

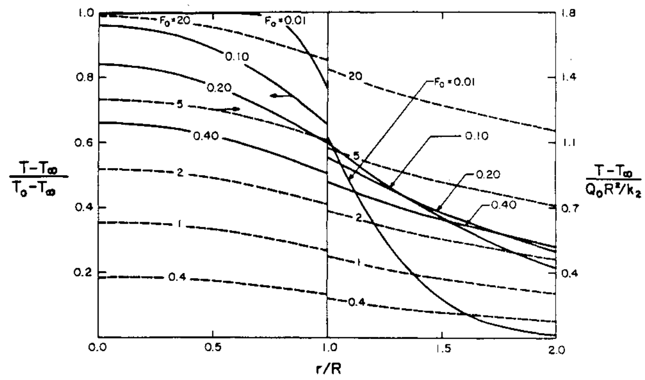

Fig. 4 Temperatures in an infinite cylinder bounded by an infinite conducting medium after a sudden change in medium temperature (—) and sudden initiation of heat generation (----). $Bi_c = 10$, $K = 1$, $A = 10$.

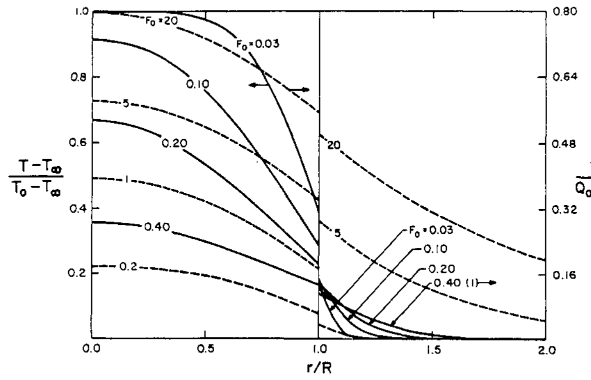

Fig. 3 Temperatures in an infinite cylinder bounded by an infinite conducting medium after a sudden change in medium temperature (—) and sudden initiation of heat generation (----). $Bi_c = 10$, $K = 1$, $A = 0.1$.

temperature T_0 in both regions, Eq. (1) is replaced by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta_1}{\partial r}\right)+\frac{Q_0}{k_1}=\frac{1}{\alpha_1}\frac{\partial \theta_1}{\partial t}, \qquad 0 \leq r \leq R \tag{11}$$

with the initial condition

$$\theta_1(r,0)=0 \tag{12}$$

All other conditions remain the same. Thus, the solution to this problem may be expressed as[1,3]

$$\theta_1(r,t)=\frac{4Q_0k_2\alpha_2}{\pi^2 R}\int_0^\infty \frac{J_1(uR)J_0(ur)[1-\exp(-\alpha_1u^2t)]}{u^4[\psi^2(u)+\theta^2(u)]}\mathrm{d}u \tag{13}$$

and

$$\theta_2(r,t)=\frac{2Q_0\sqrt{\alpha_2}}{\pi}\int_0^\infty \frac{J_0(aur)\phi(u)-Y_0(aur)\psi(u)]}{u^3[\psi^2(u)+\phi^2(u)]}J_1(uR) \times[1-\exp(-\alpha_1u^2t)]\mathrm{d}u \tag{14}$$

Evaluation and Results

Numerical evaluation of Eqs. (7) and (8) and (13) and (14) poses some challenges because of the oscillatory behavior of the integrands. The present authors have found that an effective scheme can be constructed by dividing the range of integration into intervals of progressively increasing length, and evaluating the integral over each subinterval by Romberg integration. Additional intervals are added until values of the added integrals are sufficiently small. Difficulties arise for short times in Eqs. (7) and (8), and for long times in Eqs. (13) and (14), because the strong damping effect of the exponential factor is mitigated or removed in these cases.

Typical numerical results are shown in Figs. 1-4, in which a normalized temperature difference vs Fourier number $F_0 = \alpha_1 t/R^2$ has been plotted for several values of a Biot number $Bi = h_c R/k_1$. The parameters characterizing the two media are the thermal conductivity ratio $K = k_1/k_2$ and the ratio of volumetric heat capacities, $A = K(\alpha_2/\alpha_1)$. The solid curves are for the cooling problem, without heat generation, and the dashed curves are for the heat generation problem. In general, we see that a temperature discontinuity develops at the interface and grows in the heat generation problem, but gradually disappears for conductive cooling. In Fig. 1, the effect of a large disparity in the thermal conductivities of the two media is demonstrated, which produces sharp temperature gradients in the inner cylinder, leading to large thermal stresses as described in Ref. 2. This may be contrasted with Fig. 2, where the conductivity disparity is reversed. Note the much smaller temperature discontinuity even with the same contact conductance.

In Figs. 3 and 4, the effect of widely different volumetric heat capacities is shown. This appears to have an effect similar to that of differing conductivities, but to a much smaller degree.

It is generally clear from these results that larger temperature discontinuities at the interface between the two media occur in the conductive cooling problem. These discontinuities persist as a result of the thermal contact resistance.

Conclusions

The unsteady temperature distribution in an infinite cylinder surrounded by an infinite conductive medium can be expressed in integral form via the Laplace transform method. These integrals can be numerically evaluated by Romberg integration to yield quantitative results of high accuracy.

When the infinite conductive medium has much higher conductivity than the cylinder, large temperature gradients are produced in the cylinder. The thermal contact resistance does mitigate the gradients considerably, however; quantitative characterization of this effect can be obtained through study of the figures. These figures should also be useful for checking numerical solution techniques for more geometrically complex problems.

Acknowledgments

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